Exhibit 8

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On $N = 2$ Supersymmetric QCD with 4 Flavors

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Seiberg and Witten's proposed solution of $N = 2$ SQCD with $N_c = 2$ and $N_F = 4$ is known to conflict with instanton calculations in three distinct ways. Here we show how to resolve all three discrepancies, simply by reparametrizing the elliptic curve in terms of quantities $\tau_{\text{eff}}^{(0)}$ and $\tilde{u}$ rather than τ and $u = \langle \text{Tr}\, \underset{\sim}{A}^2 \rangle$. $SL(2, \mathbb{Z})$ invariance of the curve is preserved. However, there is now an infinite ambiguity in the relation between $\tau_{\text{eff}}^{(0)}$ and τ and between $\tilde{u}$ and u, corresponding to an infinite number of unknown

coefficients in the instanton expansion. Thus the reinterpreted curve (unlike the cases $N_F < 4$) no longer determines the quantum modulus u as a function of the classical VEV a.

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1. Review of discrepancies between the Seiberg-Witten solutions and instanton calculations

The solutions of $N=2$ supersymmetric QCD with gauge group $SU(2)$ proposed by Seiberg and Witten [1] have recently been tested against first-principles instanton calculations, up to the two-instanton level [2-4]. As reviewed below, these tests have resulted in perfect agreement when N_F, the number of quark hypermultiplet flavors, is ≤ 2. On the other hand, interesting discrepancies have emerged for the cases $N_F=3$ [4] and $N_F=4$ [3]. In this paper, we present our conjecture for how to resolve these discrepancies for $N_F=4$. Just like the conjectured resolution for $N_F=3$ [5], we will not actually need to modify the elliptic curve that governs the solution (Eq. (16.35) of [1]). Instead, we will change the interpretation of the parameters τ and $\tilde{u}$ that appear in this expression; they will have a different (and highly underdetermined) relation to physical observables of the microscopic theory than that proposed by Seiberg and Witten. In particular, in sharp contrast to the cases $N_F<4$, the reinterpreted curve for $N_F=4$ no longer determines the quantum modulus

$$u \;=\; \langle \mathrm{Tr}\, \underset{\sim}{A}^2 \rangle \;, \tag{1}$$

as a function of the classical VEV a of the adjoint Higgs $\underset{\sim}{A}=A^a\tau^a/2$.

The solutions given in [1] make precise predictions for all multi-instanton contributions to the low-energy physics. The functional form of these contributions is tightly constrained by holomorphy, the $U(1)_R$ anomaly, renormalization group (RG) invariance, and the Matone relation [6-8] between the prepotential $\mathcal{F}(a)$ and the modulus $u(a)$. For $N_F>0$, additional constraints come from flavor symmetries, the decoupling limit for heavy flavors, and a discrete $\mathbb{Z}_2$ symmetry forbidding odd-instanton contributions when any quark mass vanishes. As we showed in Refs. [2,3,9,10], all these constraints are built into the instanton calculus as well.[1] Therefore, in testing the Seiberg-Witten solutions against explicit instanton calculations, one can "mod out" these constraints, and focus on a basic subset of nontrivial numerical predictions extracted from the elliptic curves at each

[1] The all-instanton-orders proof of the Matone relation given in [10] was ostensibly limited to the case $N_F=0$. However, the reader can easily verify that the proof goes through for arbitrary numbers of (massless or massive) hypermultiplets. The key point here is that Eqs. (20)-(21) of [10] can be directly extended to $N_F>0$, see Eq. (7.20) of [3]. The Matone relation can also be understood as a Ward identity for (broken) superconformal invariance [11].

order in the instanton expansion. These numbers are then compared against the results of a finite-dimensional integration over the (multi-)instanton super-moduli.

N_F	*parameter fixing*	*first nontrivial prediction of curve (≥ 1 massless quark)*	*first nontrivial prediction of curve (all $m_q > 0$)*
0	1-loop	1-inst	N/A
1	1-loop	2-inst	2-inst
2	1-loop	2-inst	2-inst
3	1-loop+2-inst [Eq. (4)]	4-inst	3-inst
4	1-loop + all-even-insts [Eqs. (6), (42)]	*none* if all $m_q = 0$; otherwise 4-inst	3-inst

Table 1. For each $N_F \leq 4$, the second column gives the order in the semiclassical expansion at which the input parameters $\Lambda_{N_F}^{\mathrm{SW}}$ and $\tilde{u}$ (or $\tau_{\mathrm{eff}}^{(0)}$ and $\tilde{u}$ for $N_F = 4$) in the Seiberg-Witten curves are fixed in terms of physical quantities in the microscopic theory. The third and fourth columns (respectively, at least one massless quark, and all quarks massive) then give the order in the semiclassical expansion at which a nontrivial numerical prediction is first made by the curve, which can be compared against an instanton calculation ("nontrivial" is defined in the text). To date, no numerical tests beyond the 2-instanton level have been performed, so that the models with $N_F = 0, 1, 2$ are on a firmer footing than the ones with $N_F = 3, 4$.

In formulating these tests, and evaluating their outcome, it is crucial to distinguish between what is *input* (i.e., the fixing of conventions) versus what is *output* (i.e., a definite numerical prediction). Since this distinction will be key to resolving the discrepancies for $N_F = 4$, we should first review the models with $N_F \leq 3$; see Table 1 for a summary. The input parameters for the elliptic curves with $0 \leq N_F \leq 3$ are the dynamically generated scale $\Lambda_{N_F}^{\mathrm{SW}}$ and the modulus $\tilde{u}$, as well as the N_F quark masses. (Notation: henceforth we will denote by $\tilde{u}$ the parameter in the Seiberg-Witten curves, and reserve the symbol u always to mean $\langle \mathrm{Tr}\, \underset{\sim}{A}^2 \rangle$.) The curve defines an implicit choice of regularization scheme

for $\Lambda^{\rm SW}_{N_F}$. For any physical quantity the n-instanton contributions are proportional to $(\Lambda^{\rm SW}_{N_F})^{n(4-N_F)}$; in order to test the specific numerical predictions of the curves against instantons, one needs to know how to relate $\Lambda^{\rm SW}_{N_F}$ to (say) the analogous quantity $\Lambda^{\rm PV}_{N_F}$ in the Pauli-Villars (PV) scheme which is most natural for instanton calculations [12]. As explained by Finnell and Pouliot [13] and reviewed in Sec. 2 below, this "dictionary" is fixed at the 1-loop level, and reads:

$$(\Lambda^{\rm SW}_{N_F})^{4-N_F} \;=\; 4(\Lambda^{\rm PV}_{N_F})^{4-N_F}\ , \qquad N_F = 0,1,2,3\ . \tag{2}$$

Similarly, for $0 \le N_F \le 2$, a discrete symmetry on the u plane combined with a semiclassical analysis of the singularities gives simply [1]

$$\tilde{u} \;=\; u\ , \qquad N_F = 0,1,2\ . \tag{3}$$

Equations (2)-(3) are the input; the predictions for $\mathcal{F}(a)$ and $u(a)$ first lie at the 1-instanton level for $N_F = 0$, and at the 2-instanton level for $N_F = 1,2$ due to the $\mathbb{Z}_2$ symmetry mentioned above.[2] The tests for $N_F = 0$ have been performed at the 1-instanton [13] and at the 2-instanton [9] level, and the tests for $N_F = 1,2$ have been carried out at the 2-instanton level [2-4], all with exact agreement.

Next we consider the case $N_F = 3$. Here Seiberg and Witten (in Sec. 14.1 of [1]) continue to assume the equality (3), even though it is no longer required by a symmetry argument. Indeed, for $N_F = 3$, the $\mathbb{Z}_{4-N_F}$ symmetry that acts on the complex u plane is trivial [1]; consequently Eq. (3) can be generalized to $\tilde{u} = u - u_0 \cdot (\Lambda^{\rm SW}_3)^2$ where u_0 is a numerical constant. (This possibility is noted by Seiberg and Witten, but not exploited.) A nonzero value for u_0 has been extracted from a 2-instanton calculation by Aoyama et al; they find [4]

$$\tilde{u} \;=\; u - u_0 \cdot (\Lambda^{\rm SW}_3)^2\ , \quad u_0 \;=\; -\frac{1}{2^4 3^3}\ , \qquad N_F = 3 \tag{4}$$

contradicting the naive assumption (3). It is natural to hypothesize [5] that the solution to the $N_F = 3$ model is still given by the Seiberg-Witten curve (Eq. (14.5) of [1]), with the substitution (4) rather than (3). In sum, for $N_F = 3$, both the 1-loop level and the

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[2] When quark masses are nonzero the models with $N_F = 1,2$ have 1-instanton contributions too; however, due to built-in heavy-quark decoupling properties, these contributions are pegged to the 1-instanton term for $N_F = 0$ and so are not independent tests [2,3].

2-instanton level should be considered as *input* due to the extra degree of freedom u_0; the first testable predictions extracted from the curve then lie at the 3-instanton level (or, if one of the quarks is massless, at the 4-instanton level).

In this paper we will focus on the particularly interesting case $N_F = 4$. Here, unlike the previous cases, the proposed solution and the instanton calculation disagree at the level of the low-energy effective $U(1)$ Lagrangian $\mathcal{L}_{\rm eff}$ [3]. In fact, this disagreement can already be seen in the massless model. In this case both the β-function and the $U(1)_R$ anomaly vanish identically, so that the microscopic coupling g_4 and θ-parameter θ_4 (which can no longer be rotated away) assemble into a single scale-independent holomorphic coupling $\tau = \frac{8\pi i}{g_4^2} + \frac{\theta_4}{\pi}$. Furthermore $\mathcal{L}_{\rm eff}^{(0)}$ is simply the Lagrangian of a classical $N = 2$ free field theory; its overall normalization, $\tau_{\rm eff}^{(0)}$, enters the BPS formulae for the dyon masses [1]. (The superscript (0) will denote the massless case.) In Section 10 of [1] Seiberg and Witten make the strong additional assumption that, thanks to the absence of a running coupling constant, the effective $U(1)$ coupling equals the classical $SU(2)$ coupling,

$$\tau_{\rm eff}^{(0)} \;=\; \tau \;, \tag{5}$$

which implies no quantum (perturbative or non-perturbative) corrections to τ. Instead, as we demonstrated in Ref. [3], a first-principles instanton calculation gives[3]

$$\tau_{\rm eff}^{(0)} \;\equiv\; \tfrac{1}{2}\mathcal{F}^{(0)\prime\prime}(a) \;=\; \tau \;+\; \frac{i}{\pi}\sum_{n=0,2,4\ldots} c_n\, q^n \;, \qquad q \;\equiv\; \exp(i\pi\tau) \;. \tag{6}$$

In particular a nonzero 2-instanton contribution $c_2 = -\frac{7}{2^6\,3^5}$ was calculated in Ref. [3]. The 1-loop perturbative constant c_0, while not considered in Ref. [3], will turn out to be crucial to our proposed resolution. We calculate below that $c_0 = 4\log 2$ in the PV scheme.[4]

A second disagreement in the $N_F = 4$ model arises when at least one of the four hypermultiplets has a nonzero mass, say $m_4 \neq 0$. In the double scaling limit defined by $m_4 \to \infty$ together with $g_4 \to 0$ in a specific way reviewed below, the heavy flavor decouples, and the model is supposed to flow to the $N_F = 3$ theory. This requires the identification

[3] Note that we use q for the 1-instanton factor rather than for the 2-instanton factor as in [1].

[4] If the classical exactness assumption (5) were correct, there would be no need to introduce a regularization scheme, since scheme dependence is a one-loop effect. As discussed in Sec. 3, although the $N_F = 4$ model is a finite theory, the microscopic $SU(2)$ coupling g_4 must still be defined in a particular scheme; that is because the finiteness is due to cancellations between individually UV divergent graphs.

of the dynamical scale $\Lambda^{\rm SW}_{N_F}$ for $N_F = 3$, with the parameters of the $N_F = 4$ theory. As explained below, working in the PV scheme and using the dictionary (2), one obtains:

$$\Lambda_3^{\rm SW} \;=\; 4\Lambda_3^{\rm PV} \;=\; 4m_4\,\exp(-8\pi^2/(g_4^{\rm PV})^2)\ , \tag{7}$$

where $g_4^{\rm PV}$ is the microscopic PV coupling in the 4-flavor model. In contrast, the relation given in [1],

$$\Lambda_3^{\rm SW} \;=\; 64m_4\,\exp(-8\pi^2/(g_4^{\rm SW})^2)\ , \tag{8}$$

involves a proportionality constant of 64 rather than 4. Again, if the quantum corrections to Eq. (5) were absent, there could be no distinction between the PV coupling constant $g_4^{\rm PV}$ and the "classical" $g_4^{\rm SW}$ of Seiberg and Witten, hence no accounting for the factor of 16 mismatch between Eqs. (7) and (8).

Thirdly, even with the identification (8), the specific $N_F = 4$ solution proposed in [1] flows to the uncorrected version of the $N_F = 3$ model which fails to incorporate the shifted definition of $\tilde{u}$ given in Eq. (4).

Our principal aim is to explain how to resolve all three of these $N_F = 4$ discrepancies in a simple way, through a reinterpretation of the quantities τ and $\tilde{u}$ that enter into the massive Seiberg-Witten curve. In particular, rather than being modular forms of the microscopic $SU(2)$ parameter τ, the coefficients of the massive curve will be functions of the effective *massless* $U(1)$ coupling $\tau_{\rm eff}^{(0)}$ defined by the all-even-instanton series (6). (Obviously the factor of 16 between Eqs. (7) and (8) will be automatically accounted for by the exponentiation of $c_0 = 4\log 2$.) This redefinition preserves the important $SL(2,\mathbb{Z})$ invariance of the elliptic curve, as well as the stringent residue condition described in Sec. 17 of [1]. However, it also introduces an infinite ambiguity into the solution, parametrized by the infinite number of as-yet-undetermined numerical coefficients c_4, c_6, $\ldots$, in Eq. (6) (a similar series, Eq. (42), relates $\tilde{u}$ and u). These issues are discussed in Sec. 3 below, which is devoted to the case $N_F = 4$, and also contains our conclusions. But first, in Sec. 2, we lay some necessary groundwork in the cases $N_F < 4$. In particular we review Weinberg's matching prescription between high- and low-energy gauge theories [14], as this formalism lies at the heart of the physics. In the process we will also specify the perturbative, one-loop contributions to the prepotential for $N_F < 4$, correcting some incomplete expressions in the literature.

The main result of this paper is the reinterpreted $N_F = 4$ massive curve which agrees with all available perturbative and multi-instanton calculations. Although the input parameters of this curve receive contributions from all even numbers of instantons, the curve

does contain definite numerical predictions at the 3-instanton level (assuming nonvanishing quark masses) that can, in principle, be tested against a semiclassical calculation. We anticipate that similar reinterpretations need to be made in the general class of $N=2$ models with gauge group $SU(N_c)$ and $N_F = 2N_c$ for which the β-function vanishes.

2. Weinberg's matching prescription and the Seiberg-Witten regularization scheme for $N_F < 4$

The physics of $N=2$ SQCD utilizes—in two distinct but equally important ways—a matching prescription between a "high-energy" and a "low-energy" gauge theory. On the one hand, the $SU(2)$ gauge group spontaneously breaks down to $U(1)$ as the adjoint Higgs $\underset{\sim}{A}$ acquires a complex VEV $\langle \underset{\sim}{A} \rangle = a\tau^3$. For energy scales $E \ll M_W = \sqrt{8}|a|$, the dynamics is governed by a nonrenormalizable Wilsonian effective action with $U(1)$ gauge invariance, formally obtained by integrating out the heavy quanta. On the other hand, one also needs to understand the RG decoupling mentioned earlier, whereby the $SU(2)$ theory with N_F flavors of quark hypermultiplets flows to the $SU(2)$ theory with $N_F - 1$ flavors, in the limit that one of the quarks becomes infinitely massive.

Both types of matching may be accomplished with the use of Weinberg's one-loop formula [14]

$$\frac{1}{g^2_{\mathrm{LE}}(\mu)} \overset{\text{1-loop}}{=} \frac{1}{g^2_{\mathrm{HE}}(\mu)} - \lambda(\mu) \ . \tag{9}$$

Here μ is the characteristic momentum scale of the light fields; $g_{\mathrm{LE}}(\mu)$ and $g_{\mathrm{HE}}(\mu)$ are renormalized gauge couplings of the low- and high-energy theories, respectively; and $\lambda(\mu)$ is a finite correction coming from one-loop contributions of heavy particles to the gauge self-energy Σ. Thus $g_{\mathrm{HE}}(\mu)$ is extracted from the complete set of one-loop contributions to Σ, whereas for $g_{\mathrm{LE}}(\mu)$ only light particles are permitted on the external and internal legs. For example, in the $\overline{\mathrm{MS}}$ scheme, $g_{\mathrm{HE}}(\mu)$ and $g_{\mathrm{LE}}(\mu)$ are defined in D dimensions in terms of the bare couplings $g_{\mathrm{HE}\,B}$ and $g_{\mathrm{LE}\,B}$ in the standard way:

$$g_{\mathrm{HE}\,B}\ \mu^{2-D/2} \ = \ g_{\mathrm{HE}}(\mu) - \ b_{\mathrm{HE}}\ g^3_{\mathrm{HE}}(\mu) \left(\frac{1}{D-4} + \tfrac{1}{2}\gamma_{\mathrm{E}} - \tfrac{1}{2}\log 4\pi \right) , \tag{10}$$

$$g_{\mathrm{LE}\,B}\ \mu^{2-D/2} \ = \ g_{\mathrm{LE}}(\mu) - \ b_{\mathrm{LE}}\ g^3_{\mathrm{LE}}(\mu) \left(\frac{1}{D-4} + \tfrac{1}{2}\gamma_{\mathrm{E}} - \tfrac{1}{2}\log 4\pi \right) , \tag{11}$$

where b_{HE} and b_{LE} are the one-loop coefficients of the corresponding β-functions. Note that when the low-energy theory is supersymmetric pure $U(1)$ gauge theory, g_{LE} receives no perturbative corrections and is scale-independent in the limit $D \to 4$.

The quantity λ has the generic form

$$\lambda(\mu) = C_1 + C_2 \log \frac{m_v}{\mu} + C_3 \log \frac{m_f}{\mu} + C_4 \log \frac{m_s}{\mu} , \tag{12}$$

where m_v, m_f and m_s are the masses of the heavy vector, fermion and scalar particles that have been integrated out. The C_i are group-theoretic constants that are tabulated by Hall (see Appendix 1 of [15], in which an error in [14] is corrected). In particular C_1 and C_2 appear only when the heavy vector particles have been integrated out; similarly, C_3 appears when there are heavy fermions, and C_4 corresponds to the heavy scalars.

The original calculations of [14-15] were performed in the dimensional regularization with $\overline{\mathrm{MS}}$ scheme (DREG). However, they can be easily translated into the supersymmetry-preserving dimensional reduction with $\overline{\mathrm{MS}}$ scheme (DRED) [16], or into the Pauli-Villars (PV) scheme. The one-loop relations between the coupling constants in different schemes for the $SU(N_{\mathrm{c}})$ gauge group can be found for example in [17,13]:

$$\frac{1}{g^2_{\mathrm{PV}}(\mu)} = \frac{1}{g^2_{\mathrm{DRED}}(\mu)} = \frac{1}{g^2_{\mathrm{DREG}}(\mu)} + \frac{1}{48\pi^2} N_{\mathrm{c}} , \tag{13}$$

independently of N_F. It turns out that for the case of spontaneous symmetry breaking $[SU(2), N_F] \longrightarrow [U(1), 0]$ considered in detail presently,

$$C_1^{\mathrm{PV}} = C_1^{\mathrm{DRED}} = 0 . \tag{14}$$

In contrast, in the case of the heavy flavor decoupling $[SU(2), N_F] \longrightarrow [SU(2),\, N_F - 1]$, vector particles do not decouple and $C_1 \equiv 0$ by definition.

Now let us apply this formalism, in turn, to these two cases of interest.

1. Spontaneous symmetry breaking

We consider the Coulomb branch of $N = 2$ SQCD, in which only the adjoint Higgs $\underset{\sim}{A} \equiv A^a \tau^a/2$ acquires a VEV, say in the τ^3 direction: $\langle \underset{\sim}{A} \rangle = a\tau^3$. (We have adopted here the VEV normalization conventions of [1]; the translation formulae to the original normalization of [18] used in our previous work are assembled in Appendix A.) The $SU(2)$ component of the adjoint $N = 2$ supermultiplet that is parallel to the VEV then remains massless, while the components $\propto\ \tau^1$ or τ^2 acquire a mass $M_W = \sqrt{8}|a|$. In addition there are $2N_F$ quark multiplets Q_f and $\tilde{Q}_f$, $f = 1, \cdots, N_F$, in the fundamental representation of the gauge group. The '1' and '2' color components of these multiplets acquire masses

$|\sqrt{2}a + m_f|$ and $|\sqrt{2}a - m_f|$, respectively, as can be seen from a tree-level examination of the $N = 2$ invariant superpotential [1]

$$\mathcal{W} \ = \ \sum_{f=1}^{N_F} \sqrt{2}\tilde{Q}_f \underset{\sim}{\Phi} Q_f + m_f \tilde{Q}_f Q_f \ . \tag{15}$$

Here $\underset{\sim}{\Phi}$ is the $N = 1$ adjoint chiral superfield whose lowest component is $\underset{\sim}{A}$; color indices are suppressed.

In the PV or DRED schemes one then has

$$\lambda(\mu) \ = \ C_{\text{adj}} \log \frac{M_W}{\mu} \ + \ C_{\text{fund}} \sum_{f=1}^{N_F} \big(\log \frac{|\sqrt{2}a + m_f|}{\mu} + \log \frac{|\sqrt{2}a - m_f|}{\mu} \big) \ , \tag{16}$$

where C_{adj} and C_{fund} are numerical constants. Setting $\mu = M_W$ for simplicity and extracting $C_{\text{fund}} = 1/16\pi^2$ from Ref. [15], one finds

$$\lambda(M_W) \ = \ \frac{1}{16\pi^2} \sum_{f=1}^{N_F} \log \left| \frac{2a^2 - m_f^2}{8a^2} \right| \ , \tag{17}$$

so that Eq. (9) becomes

$$\frac{1}{g_{\text{eff}}^2} \ \stackrel{\text{1-loop}}{=} \ \frac{4 - N_F}{8\pi^2} \log\left(\frac{M_W}{\Lambda_{N_F}} \right) \ + \ \frac{N_F}{8\pi^2} \log 2 \ - \ \frac{1}{16\pi^2} \sum_{f=1}^{N_F} \ \log \left| 1 - m_f^2/2a^2 \right| . \tag{18}$$

In this expression $g_{\text{eff}} \equiv g_{\text{LE}}$ denotes the effective $U(1)$ coupling constant, Λ_{N_F} is the PV or equivalently the DRED dynamical scale (we drop the PV superscript henceforth)

$$\Lambda_{N_F}^b \ = \ \mu^b \exp[-8\pi^2/g_{\text{PV}}^2(\mu)] \ = \ \mu^b \exp[-8\pi^2/g_{\text{DRED}}^2(\mu)] \ , \tag{19}$$

and we have used the fact that the (negated) coefficient of the β-function for these models is $b = 2N_c - N_F = 4 - N_F$. We make the following comments:

(i) Equations (18)-(19) extend to $N_F \geq 0$ the case of $N_F = 0$ considered by Finnell and Pouliot, who obtained simply [13]

$$\frac{1}{g_{\text{eff}}^2} \ \stackrel{\text{1-loop}}{=} \ \frac{1}{g_{\text{PV}}^2(M_W)} \ = \ \frac{1}{2\pi^2} \log\left(\frac{M_W}{\Lambda_0} \right) \ . \tag{20}$$

This is referred to as the absence of threshold corrections.

(ii) As usual in a supersymmetric chiral theory [19], $1/g_{\rm eff}^2$ is the imaginary part of a holomorphic complexified coupling constant

$$\tau_{\rm eff} \;=\; \frac{8\pi i}{g_{\rm eff}^2} + \frac{\theta_{\rm eff}}{\pi} \;. \tag{21}$$

This implies that Eq. (18) may be analytically continued away from the imaginary axis, as follows:

$$\tau_{\rm eff} \;\stackrel{\text{1-loop}}{=}\; (i/\pi)(4-N_F)\log\!\left(\frac{\sqrt{8}a}{\Lambda_{N_F}}\right) \;+\; \frac{iN_F}{\pi}\log 2 \;-\; \frac{i}{2\pi}\sum_{f=1}^{N_F}\log\left(1-m_f^2/2a^2\right) \;. \tag{22}$$

2. Heavy flavor decoupling

Next we consider the case in which a single hypermultiplet flavor becomes very heavy (say, $m_{N_F}\to\infty$) and decouples from the spectrum, leaving behind $N=2$ SQCD with one fewer flavor. Choosing $\mu = m_{N_F}$ we find that $\lambda(m_{N_F})=0$ as follows from Eq. (12) with $C_1 = C_2 \equiv 0$; consequently

$$g_{N_F}^{-2}(m_{N_F}) \;=\; g_{N_F-1}^{-2}(m_{N_F}) \tag{23}$$

in the PV or DRED schemes. Rewriting this relation in an RG-invariant way using (19), one obtains

$$m_{N_F}\cdot\Lambda_{N_F}^{4-N_F} \;=\; \Lambda_{N_F-1}^{4-(N_F-1)} \;. \tag{24}$$

The appropriate double scaling limit is therefore defined by $m_{N_F}\to\infty$ and $\Lambda_{N_F}\to 0$ with the product on the left-hand side of (24) being held fixed.

In the remainder of this section we will relate the dynamical scales $\Lambda_{N_F}^{\rm SW}$ that appear in the Seiberg-Witten elliptic curves, on the one hand, to the PV or DRED scales (19), on the other hand. This is done by comparing Eqs. (22)-(24) to the explicit solutions proposed in [1].

3. Relating the Seiberg-Witten scheme to the PV or DRED schemes

The Seiberg-Witten elliptic curve for the $N_F = 0$ theory is [1]

$$y^2 \;=\; x^2(x-u) \;+\; \frac{1}{4}(\Lambda_0^{\rm SW})^4 x \;. \tag{25}$$

The curve defines a dynamical scale $\Lambda_0^{\rm SW}$ in a particular scheme, the "Seiberg-Witten scheme"; the superscript SW is introduced to distinguish it from Λ_0 in the PV or DRED

schemes. The correspondence between the SW and the PV schemes for $N_F = 0$ has been examined by Finnell and Pouliot [13], using Weinberg's matching formula. From the elliptic curve (25), they calculate $\tau_{\rm eff} \simeq (2i/\pi)\log\left(8u/(\Lambda_0^{\rm SW})^2\right)$ valid in the semiclassical regime, $u \simeq 2a^2$. Comparing this result with Eq. (22) with $N_F = 0$ then yields [13]

$$(\Lambda_0^{\rm SW})^4 \;=\; 4\,\Lambda_0^4\;. \tag{26}$$

While this derivation was ostensibly performed at the one-loop level, it is well known that such relations between Λ's defined in different renormalization schemes are actually one-loop exact [20], regardless of the presence of supersymmetry. (In contrast, the definition (19) of the dynamical scale itself is one-loop exact under the RG of the Wilsonian effective action, but only because of supersymmetry [19,21].)

Next we consider the cases $0 < N_F < 4$. Let us introduce the symmetric polynomials in the masses:

$$M_0^{(N_F)} = 1\,,\quad M_1^{(N_F)} = \sum_{i=1}^{N_F} m_i^2\,,\quad M_2^{(N_F)} = \sum_{i<j}^{N_F} m_i^2 m_j^2\;\;,\;\;\cdots\;\;,\;\; M_{N_F}^{(N_F)} = \prod_{j=1}^{N_F} m_j^2\,. \tag{27}$$

The curves are then given by [1][5]

$$y^2_{(N_F)} \;=\; x^2(x-\tilde{u}) + \frac{1}{4}\sqrt{M_{N_F}^{(N_F)}}\,(\Lambda_{N_F}^{\rm SW})^{4-N_F}\,x - \frac{1}{64}(\Lambda_{N_F}^{\rm SW})^{8-2N_F}\sum_{\delta=0}^{N_F-1} M_\delta^{(N_F)}(x-\tilde{u})^{N_F-1-\delta} \tag{28}$$

Seiberg and Witten simply equate $\tilde{u} \equiv u$ for $N_F = 1,2,3$ but, as reviewed earlier, for $N_F = 3$ this should be corrected to Eq. (4). It is easily checked that in the decoupling limit $m_{N_F} \to \infty$, one obtains the desired result $y^2_{(N_F)} \to y^2_{(N_F-1)}$ if and only if one makes the identification

$$m_{N_F}\cdot(\Lambda_{N_F}^{\rm SW})^{4-N_F} \;=\; (\Lambda_{N_F-1}^{\rm SW})^{4-(N_F-1)}\;. \tag{29}$$

[5] A technical aside: In the absence of quark masses we have identified alternative curves for both $N_F = 2$ and $N_F = 3$ with the desired singularity structure. They are $y^2 = x^2(x-u) - \frac{9}{64}(\Lambda_2^{\rm SW})^4(x-u/9)$ and $y^2 = x^2(x-u) - \frac{1}{64}(\Lambda_3^{\rm SW})^2u^2$, respectively. Consistent with the physical arguments in [1], in the former case the discriminant $\Delta(u)$ correctly factors into a product of two double roots, while in the latter case $\Delta(u)$ is the product of a simple root with a quartic root. However, in each case $\Delta(u)$ has the wrong singularity structure once mass terms are added, which can be seen most easily in the case that all masses are set equal.

Notice that this is the same recursion relation as for the PV and DRED schemes, Eq. (24). From the $N_F = 0$ "boundary condition" (26), one immediately derives the dictionary between schemes given in Eq. (2) above.

Before passing to the case $N_F = 4$, we note that the perturbative, one-loop structure of the effective $U(1)$ coupling contained in Eq. (22) has rarely appeared correctly in the literature. This expression has recently been confirmed by [22]; rather than invoke the Weinberg prescription, these authors extract the result directly from the elliptic curves, for arbitrary N_c.

3. Resolving the discrepancies in the $N_F = 4$ solution

We now turn to the interesting case $N_F = 4$. In this model the β-function vanishes and no dynamical scale is generated. Note that it is trivial to extend the RG matching relation (24) to this case. The relation (23) still holds when $\mu = m_4 \to \infty$; consequently

$$m_4 \exp(-8\pi^2/g_4^2) \;=\; \Lambda_3 \tag{30}$$

using Eq. (19). Here, and henceforth, $g_4(\mu) \equiv g_4$ is the scale-independent microscopic coupling in the PV or DRED scheme[6]; it combines with θ_4 to form a single scale-independent PV or DRED holomorphic parameter

$$\tau \;=\; \frac{8\pi i}{g_4^2} + \frac{\theta_4}{\pi} \;. \tag{31}$$

Before turning to the elliptic curve, let us revisit the three discrepancies with instanton physics that we wish to address. The first is the discrepancy between Eqs. (5) and (6) which relate $\tau_{\rm eff}^{(0)}$ and τ. (As above, we will use the superscript (0) to denote the massless case.) The one-loop constant c_0 in Eq. (6) was not considered in Ref. [3]. In light of the

[6] One may ask why, in this finite theory, it is nevertheless necessary to specify a scheme. As mentioned earlier, this is because the finiteness is due to an "∞ minus ∞" cancellation between divergent diagrams, which is intrinsically ill defined. For this reason we cannot simply construct physical quantities directly from the bare tree-level coupling g_B, as would be natural to do if all individual graphs converged. That g_B cannot be a scheme-independent physical quantity can be seen from Eq. (10), which applies equally to the DRED and DREG schemes, and which would then imply for $N_F = 4$: $g_{\rm DRED}(\mu) = g_{\rm DREG}(\mu) = g_B\, \mu^{2-D/2}$ in contradiction to Eq. (13).

preceding discussion, we can now read off the value $c_0 = 4\log 2$ from Eq. (22), which is easily extended to the case $N_F = 4$:

$$\tau_{\text{eff}} \;\equiv\; \tfrac{1}{2}\mathcal{F}''(a) \;=\; \tau \;+\; \frac{4i}{\pi}\log 2 \;-\; \frac{i}{2\pi}\sum_{f=1}^{4} \log\left(1 - m_f^2/2a^2\right) \;+\; \mathcal{O}(q)\;. \tag{32}$$

An independent derivation of this 1-loop relation (which tests our proposed reinterpretation of the massive curve) is discussed below.

The second and third discrepancies involve properties of the massive $N_F = 4$ curve (Sec. 16.3 of [1]). Consider the RG decoupling property when the quark mass m_4 grows large. In the double scaling limit the curve indeed collapses to the $N_F = 3$ curve (28); however, as reviewed in Appendix B, this limiting behavior requires the identification [1]

$$64\, m_4 \exp\left(-8\pi^2/(g_4^{\text{SW}})^2\right) \;=\; \Lambda_3^{\text{SW}} \;=\; 4\Lambda_3 \tag{33}$$

where the second equality follows from Eq. (2). This apparently contradicts Eq. (30) above; more precisely it means that the coupling g_4^{SW} used in [1] cannot be the expected DRED quantity g_4. And thirdly, the $N_F = 3$ curve that one flows to in this way has $\tilde{u} = u$ rather than the shifted definition (4).

As the reader can anticipate, we will posit that the parameter τ that appears pervasively in the Seiberg-Witten curve for $N_F = 4$ should really be identified with the effective massless $U(1)$ coupling $\tau_{\text{eff}}^{(0)}$, Eq. (6), rather than with the microscopic $SU(2)$ coupling τ, Eq. (31). By definition, this reinterpretation resolves the first of the three discrepancies. Pleasingly it also resolves the second discrepancy: the constant factor $c_0 = 4\log 2$, when exponentiated, precisely compensates for the factor of 16 mismatch between Eqs. (33) and (30). (A more stringent test of our proposal is discussed below.) Finally, the third discrepancy will be resolved by altering the relation proposed in [1] between $\tilde{u}$ and u; however, there is an infinite ambiguity in this procedure that we do not know how to eliminate.

We now review the Seiberg-Witten curve (Sec. 16 of [1]), starting with the massless case:

$$\left(y^{(0)}\right)^2 \;=\; x^3 - \tfrac{1}{4}g_2(\tau_{\text{SW}})\, x\tilde{u}^2 - \tfrac{1}{4}g_3(\tau_{\text{SW}})\,\tilde{u}^3 \;=\; W_1^{(0)}W_2^{(0)}W_3^{(0)}\;, \quad W_i^{(0)} = x - e_i(\tau_{\text{SW}})\,\tilde{u}\;. \tag{34}$$

Here g_2 and g_3 are rescaled Eisenstein series. The cubic roots e_i may be defined in terms of θ-functions; they have the semiclassical expansion

$$e_1(\tau) = \tfrac{2}{3} + 16q^2 + 16q^4 + \mathcal{O}(q^6)\;, \quad e_2(\tau) = -\tfrac{1}{3} - 8q - 8q^2 - 32q^3 - 8q^4 + \mathcal{O}(q^5)\;, \tag{35}$$

with $e_3 = -e_1 - e_2$. As noted in [1], the e_i are not strictly speaking modular forms of $SL(2, \mathbb{Z})$. Rather, they are weight-two modular forms of three different conjugate subgroups of $SL(2, \mathbb{Z})$; under the action of the full group they permute amongst themselves. The curve (34) is well known in the math literature [23]. It is designed so that if the VEVs a and a_D are extracted in the standard way as periods of the curve [1],

$$\frac{da}{d\tilde{u}} = \frac{\sqrt{2}}{8\pi} \int_{\gamma_1} \frac{dx}{y^{(0)}} , \qquad \frac{da_D}{d\tilde{u}} = \frac{\sqrt{2}}{8\pi} \int_{\gamma_2} \frac{dx}{y^{(0)}} , \tag{36}$$

then one has simply

$$a = \sqrt{\tilde{u}/2} , \qquad a_D = \tau_{\rm SW}\, a . \tag{37}$$

As expected, these are the defining equations of a classical free field theory, with $\mathcal{F}^{(0)}(a) = \tau_{\rm SW}\, a^2$ and $\tilde{u} = 2a^2$.

Seiberg and Witten make the two further assumptions $\tau_{\rm SW} = \tau$ and $\tilde{u} = u$. The first of these assumptions contradicts Eq. (6); instead, we will assume $\tau_{\rm SW} = \tau_{\rm eff}^{(0)}$ as explained above. As pointed out in [5], this, in turn, invalidates the second assumption as well; instead, one must take

$$\tilde{u} = u \cdot \left(d\tau_{\rm eff}^{(0)} / d\tau \right)^{-1} . \tag{38}$$

This latter redefinition is specific to the massless model; it follows directly from the instanton version [24,10,5] of Matone's relation, which for four massless flavors reads,

$$u = 2\pi i q \frac{\partial \mathcal{F}^{(0)}}{\partial q} , \tag{39}$$

combined with Eq. (37).

We turn finally to the massive curve. Setting $e_{ij} = e_i - e_j$, one has [1]:

$$y^2 = W_1 W_2 W_3 + e_{12} e_{23} e_{31} \left(W_1 T_1 e_{23} + W_2 T_2 e_{31} + W_3 T_3 e_{12} \right) - e_{12}^2 e_{23}^2 e_{31}^2 N . \tag{40}$$

Here $W_i = W_i^{(0)} - e_i^2\, R$, where R, N and T_i are symmetric polynomials in the four masses:

$$\begin{aligned} R &= \frac{1}{2} \sum_i m_i^2 , \qquad N = \frac{3}{16} \sum_{i>j>k} m_i^2 m_j^2 m_k^2 - \frac{1}{96} \sum_{i \neq j} m_i^2 m_j^4 + \frac{1}{96} \sum_i m_i^6 , \\ T_1 &= \frac{1}{12} \sum_{i>j} m_i^2 m_j^2 - \frac{1}{24} \sum_i m_i^4 , \quad T_2 = -\frac{1}{2} \prod_i m_i - \frac{1}{24} \sum_{i>j} m_i^2 m_j^2 + \frac{1}{48} \sum_i m_i^4 \end{aligned} \tag{41}$$

with $T_3 = -T_1 - T_2$. Seiberg and Witten observe that the coefficients of this curve are $SL(2, \mathbb{Z})$ modular invariants, provided that the aforementioned permutations amongst the

e_i are accompanied by the same permutations acting on the T_i (this is referred to as $SO(8)$ triality). They view this as strong circumstantial evidence that the dyon spectrum itself is $SL(2,\mathbb{Z})$ invariant.

In restoring agreement with instanton calculations we need not tamper with these important $SL(2,\mathbb{Z})$ properties. Rather, we will reinterpret the intrinsic parameters of the curve. In particular we will continue to take $e_i \equiv e_i(\tau_{\rm eff}^{(0)})$ where $\tau_{\rm eff}^{(0)}$ is given by the instanton series (6). It remains only to relate the parameter $\tilde{u}$ (which enters through the W_i) to the physical quantum modulus $u = \langle {\rm Tr}\, \underset{\sim}{A}^2 \rangle$. Dimensional analysis, $O(4)$ symmetry, smoothness in the masses, and the above-given massless limit suggest the following generic relation:

$$\tilde{u} \;=\; u \cdot \Big(\frac{d\tau_{\rm eff}^{(0)}}{d\tau}\Big)^{-1} \;+\; R \cdot \sum_{n=0,2,4\cdots} \alpha_n\, q^n \;, \quad q = \exp(i\pi\tau) \;. \tag{42}$$

(This expression is to be compared with the Seiberg-Witten proposal [1]

$$\tilde{u} \;=\; u \;-\; \tfrac{1}{2} e_1(\tau) R \tag{43}$$

which is already faulty in the massless limit.) The absence of odd instanton contributions is due to the discrete $\mathbb{Z}_2$ symmetry discussed earlier, the mass parameter R being even under this symmetry. The numerical coefficients α_n in (42) may be constrained using a variety of physics considerations explained in Appendix B. We find that $\alpha_0 = -1/3$, and $\alpha_2 = 37/(3^3\, 2^5)$; the latter value disagrees again with the proposed relation (43), but is necessary to recapture the shifted definition (4) of $\tilde{u}$ in the decoupling limit.

We conclude with the following comments:

(i) The higher-instanton contributions α_n with $n = 4, 6, \cdots$ remain completely undetermined, as do the constants c_n with $n = 4, 6, \cdots$ in the relation (6) between $\tau_{\rm eff}^{(0)}$ and τ. When masses are incorporated into Eq. (6), Matone's relation [6,7,10] may give an interesting correspondence between the two series (as it already does in the massless case [5]).

(ii) Our lack of complete knowledge of the relation between $\tilde{u}$ and u does not actually affect the low-energy effective $U(1)$ Lagrangian. This is because a and a_D are still determined by Eqs. (36) (with y instead of $y^{(0)}$ in the massive case). Both sides of these equalities involve the (independent) variable $\tilde{u}$ rather than the unknown (dependent) variable u. The former is then eliminated in favor of a, giving a prepotential in which neither u nor $\tilde{u}$ appears: $\mathcal{F} = \mathcal{F}\big(a, \tilde{u}(a); \{m_i\}\big) \equiv \mathcal{F}(a; \{m_i\})$. In other words, so long as the classical VEV a is considered the independent variable (and not the quantum modulus

$u = \langle \mathrm{Tr}\, \underset{\sim}{\Phi}^2 \rangle$ on which a depends in a presently unknown way, and *vice versa*), the prepotential is "known" as a function of $\tau_{\rm eff}^{(0)}$ (albeit not as a function of the microscopic τ). In contrast, for $N_F < 4$, both a and $\mathcal{F}$ are known functions of u as well.

(iii) As a stringent test of our proposed redefinitions $e_i \equiv e_i(\tau_{\rm eff}^{(0)})$, we have in fact constructed $\mathcal{F}(a)$ as outlined in **(ii)**, and verified that the right-hand side of Eq. (32) is indeed reproduced by the curve (paralleling Ref. [22]).

(iv) Finally we reiterate the point that the massive $N_F = 4$ curve is still $SL(2,\mathbb{Z})$ invariant, but only in terms of the rather non-intuitive quantity $\tau_{\rm eff}^{(0)}$ (the effective coupling in the *massless* theory) rather than the microscopic coupling τ. While the relation (6) between these parameters is currently unknown beyond the 2-instanton level, it would be pleasing if, in the end, the model turned out to be modular invariant in terms of τ as well.

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Appendix A. Note on conventions

Here we provide the dictionary between the normalization conventions of Ref. [1] which we have adopted in this paper, and those of Ref. [18] which we used in previous work [9,3].

The original unrescaled VEV definitions of [18] are:

$$\langle \underset{\sim}{A} \rangle = \mathrm{v}\tau^3/2 \ , \quad \langle \underset{\sim}{A}_D \rangle = \mathrm{v}_D\tau^3/2 \ , \quad \mathrm{v}_D = \frac{\partial \mathcal{F}(\mathrm{v})}{\partial \mathrm{v}} \ .$$

$$\tau_{\rm eff} \equiv \frac{8\pi i}{g_{\rm eff}^2} + \frac{\theta_{\rm eff}}{\pi} = 2\frac{\partial \mathrm{v}_D}{\partial \mathrm{v}} = 2\frac{\partial^2 \mathcal{F}(\mathrm{v})}{\partial \mathrm{v}^2} \ .$$

The VEV normalizations of [1] adopted in this paper are:

$$a = \tfrac{1}{2}\mathrm{v} \ , \qquad a_D = \mathrm{v}_D = \tfrac{1}{2}\frac{\partial \mathcal{F}(a)}{\partial a}$$

$$\tau_{\rm eff} = \frac{\partial a_D}{\partial a} = \tfrac{1}{2}\frac{\partial^2 \mathcal{F}(a)}{\partial a^2}$$

$$\mathrm{v} = \sqrt{2u} + \cdots \ , \qquad a = \tfrac{1}{2}\sqrt{2u} + \cdots$$

The prepotential for $N_F < 4$ massless flavors in the two different normalizations reads:

$$\mathcal{F}^{(N_F)}(a,\Lambda^{\rm SW}_{N_F}) = \mathcal{F}^{(N_F)}_{\rm pert} - \frac{i}{\pi}\sum_{n=2,4,6,\ldots} F^{(N_F)}_n \left(\frac{\Lambda^{\rm SW}_{N_F}}{a}\right)^{n(4-N_F)} a^2$$

$$\mathcal{F}^{(N_F)}(\mathrm{v},\Lambda_{N_F}) = \mathcal{F}^{(N_F)}_{\rm pert} - \frac{i}{\pi}\sum_{n=2,4,6,\ldots} \mathcal{F}^{(N_F)}_n \left(\frac{\Lambda_{N_F}}{\mathrm{v}}\right)^{n(4-N_F)} \mathrm{v}^2 \ .$$

Using Eq. (2), the relation between the instanton coefficients is determined to be:

$$\mathcal{F}^{(N_F)}_n = 2^{n(6-N_F)-2}\, F^{(N_F)}_n \ .$$

Appendix B. Constraints on the relation between $\tilde{u}$ and u

In this appendix we constrain some of the *a priori* unknown numerical constants α_n that relate $\tilde{u}$ to u as per Eq. (42). We use the following two considerations:

(i) Closely following Sec. 16.3 of [1], we first consider the illuminating special case $(m_1,m_2,m_3,m_4) = (0,0,m,m)$. Then $N = T_i = 0$ and one has simply $y^2 = W_1W_2W_3$ with $W_i = x - e_i\tilde{u} - e_i^2m^2$. As always, the critical points on the quantum moduli space are the values of u for which two of the three x roots coincide. In the present instance this means $e_i\tilde{u} + e_i^2m^2 = e_j\tilde{u} + e_j^2m^2$, or equivalently since $\sum e_i = 0$:

$$\tilde{u} = \{e_1m^2, e_2m^2, e_3m^2\} \simeq \{\tfrac{2}{3}m^2\,,\, (-\tfrac{1}{3} - 8q)m^2\,,\, (-\tfrac{1}{3} + 8q)m^2\} \qquad \text{(B.1)}$$

using Eq. (35). We have dropped terms of order q^2 as these do not survive the double scaling limit. Now consider the decoupling limit $m \to \infty$. On the one hand, one expects a perturbative singularity at $u \simeq 2a^2 \simeq m^2$ which corresponds, physically, to two quark multiplets becoming massless (see Eq. (32)). On the other hand, the leftover model after the two heavy flavors decouple is the massless $N_F = 2$ theory, which has singularities at $u = \pm\frac{1}{8}(\Lambda_2^{\rm SW})^2$ (see Eq. (28)). In sum,

$$u \simeq \{m^2\,,\, -\tfrac{1}{8}(\Lambda_2^{\rm SW})^2\,,\, \tfrac{1}{8}(\Lambda_2^{\rm SW})^2\} \ . \qquad \text{(B.2)}$$

Equating (B.2) with (B.1) forces $\alpha_0 = -1/3$; and furthermore $8q^{(0)}_{\rm eff}\, m^2 = \frac{1}{8}(\Lambda_2^{\rm SW})^2$ which is precisely consistent with the DRED recursion relations (29), (30) and (33) (with $g_4^{\rm SW} \equiv g^{(0)}_{\rm eff}$).

(ii) Returning to the case of four generic masses, we next consider the double scaling limit $m_4 \to \infty$, $q_{\rm eff}^{(0)} \to 0$ with the product $m_4\, q_{\rm eff}^{(0)}$ fixed at $\Lambda_3^{\rm SW}/64$ as per Eq. (33). Naively, we would like the curve (40) to collapse to the $N_F = 3$ curve defined by Eqs. (28) and (4). Instead, in this limit the right-hand side of (40) diverges badly. However we can exploit the fact that the variable x is just a dummy of integration (see Eq. (36)); an appropriate shift in x eliminates this divergence and guarantees a smooth RG limit. Accordingly we let

$$x \longrightarrow x \;+\; \big(\tfrac{2}{9} + (\tfrac{1}{32} - \tfrac{1}{3}\alpha_2)\, q^2 + \mathcal{O}(q^4)\big)\, R \;-\; \big(\tfrac{1}{3} + \mathcal{O}(q^2)\big)\, u \tag{B.3}$$

which obscures the $SL(2,\mathbb{Z})$ properties of the curve, but preserves the $\mathbb{Z}_2$ properties. The explicit factors in (B.3) have the following genesis: the constant $2/9$ eliminates the large-mass divergence; the constant $1/3$ guarantees that the cubic terms in x and u will have precisely the form $x^2(x-u)$ dictated by Eq. (28); and finally the factor $(1/32 - \alpha_2/3)$ eliminates the $(\Lambda_3^{\rm SW})^3 m_1 m_2 m_3$ term that otherwise generically appears, again in order to harmonize with Eq. (28). (This term is not forbidden by any symmetry; its absence from the $N_F = 3$ curve (28) is convention dependent, as is the form of the cubic term.) Substituting Eq. (B.3) into the $N_F = 4$ curve (40) and taking the double scaling limit then indeed reproduces the $N_F = 3$ curve (28), provided that

$$-(1 + 32\alpha_2)\, 2^{-10} \;=\; u_0\ . \tag{B.4}$$

From the value of u_0 quoted in Eq. (4) one deduces $\alpha_2 \;=\; 37/(3^3\, 2^5)$.

References

[1] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, Nucl. Phys B431 (1994) 484 , hep-th/9408099

[2] N. Dorey, V.V. Khoze and M.P. Mattis, *A two-instanton test of the exact solution of $N = 2$ supersymmetric QCD*, hep-th/9607066, Phys. Lett. B388 (1996) 324.

[3] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-instanton calculus in $N = 2$ supersymmetric gauge theory. II. Coupling to matter*, hep-th/9607202, Phys. Rev. D (in press, 15 December 1996).

[4] H. Aoyama, T. Harano, M. Sato and S.Wada, *Multi-instanton calculus in N=2 supersymmetric QCD*, hep-th/9607076.

[5] T. Harano and M. Sato, *Multi-instanton calculus versus exact results in N=2 supersymmetric QCD*, hep-th/9608060

[6] M. Matone, *Instantons and recursion relations in $N = 2$ SUSY gauge theory* Phys. Lett. B357 (1995) 342, hep-th/9506102.

[7] T. Eguchi and S.K. Yang, *Prepotentials of $N = 2$ supersymmetric gauge theories and soliton equations*, hep-th/9510183;
G. Bonelli and M. Matone, Phys. Rev. Lett. 76 (1996) 4107, hep-th/9602174;
J. Sonnenschein, S. Theisen and S. Yankielowicz, Phys. Lett. 367B (1996) 145, hep-th/9510129.

[8] G. Bonelli, M. Matone and M. Tonin, *Solving $N = 2$ SYM by the reflection symmetry of quantum vacua*, hep-th/9610026.

[9] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-instanton calculus in $N = 2$ supersymmetric gauge theory*, hep-th/9603136, Phys. Rev. D54 (1996) 2921.

[10] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-instanton check of the relation between the prepotential $\mathcal{F}$ and the modulus u in $N = 2$ SUSY Yang-Mills theory*, hep-th/9606199, Phys. Lett. B389 (in press, 26 December 1996).

[11] P.S. Howe and P. West, *Superconformal Ward identities and $N = 2$ Yang-Mills theory*, hep-th/9607239.

[12] G. 't Hooft, Phys. Rev. D14 (1976) 3432; ibid. D18 (1978) 2199.

[13] D. Finnell and P. Pouliot, *Instanton calculations versus exact results in 4 dimensional SUSY gauge theories*, Nucl. Phys. B453 (95) 225, hep-th/9503115.

[14] S. Weinberg, Phys. Lett. 91B (1980) 51.

[15] L. Hall, Nucl. Phys. B178 (1981) 75.

[16] W. Siegel, Phys. Lett. 84B (1979) 193.

[17] S.P. Martin, M.T. Vaughn, Phys. Lett. B318 (1993) 331.

[18] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nucl. Phys. B426 (1994) 19, (E) B430 (1994) 485 hep-th/9407087

[19] M.A. Shifman and A.I. Vainshtein, Nucl. Phys. B277 (1986) 456, Nucl. Phys. B359 (1991) 571.

[20] W. Celmaster and R. Gonsalves, Phys. Rev. D20 (1979) 1420

[21] M. Dine and Yu. Shirman, Phys. Rev. D50 (1994) 5389.

[22] E. D'Hoker, I. M. Krichever and D. H. Phong, *The effective prepotential of $N = 2$ supersymmetric $SU(N)$ gauge theories*, hep-th/9609041

[23] See for example N. Koblitz, *Introduction To Elliptic Curves And Modular Forms* (Springer-Verlag, 1984), Sec. I.6.

[24] F. Fucito and G. Travaglini, *Instanton calculus and nonperturbative relations in $N = 2$ supersymmetric gauge theories*, ROM2F-96-32, hep-th/9605215.